On the functional equation of generalized pseudomediality

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Abstract

Based on $D_{i,j}$ and D_{i-j} conditions (see [1]) we give a direct and straightforward method to solve the functional equation of generalized pseudomediality on quasigroups.

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In this paper we solve the functional equation of generalized pseudomediality on quasigroups

$$\alpha_1(\alpha_2(x,y),\alpha_3(z,u)) = \alpha_4(\alpha_5(\alpha_6(x,z),y)u) \tag{1}$$

where x, y, z, u are taken from an arbitrary set A and α_i are quasigroup operations on A.

Sokhats'kyi [3] proved that every quadratic parastrophically uncancelable functional equation for four object variables is parastrophically equivalent to the functional equation of mediality (for this equation see [2]) or the functional equation of pseudomediality. He solved this equation using the functional equation of generalized associativity (for this equation see [2]).

The method we used is an example of the application of our results developed in [1].

Let $\alpha_1, \dots, \alpha_6$ be a six quasigroup operations on A and forming a solution of the functional equation (1). We define $\alpha : A^4 \to A$ by

$$\alpha(x,y,z,u) = \alpha_1(\alpha_2(x,y),\alpha_3(z,u)) = \alpha_4(\alpha_5(\alpha_6(x,z),y),u) \tag{2}$$

It is obvious that (A,α) is 4-quasigroup and conditions $D_{1,2}$, $D_{3,4}$ and D_{1-3} are fulfilled in (A,α) (see [1]).

Theorem 1. *In a 4-loop (A,α) condition $D_{1,2}\&D_{3,4}\&D_{1-3}$ holds iff $\alpha(x,y,z,u) = yxzu$ where $(A,\cdot)$ is a binary group.*

Proof. Suppose that in 4-loop (A,α) condition $D_{12}\&D_{3,4}\&D_{1-3}$ holds and let e be a unit in this loop. We define

$$x \cdot y = \alpha(e,e,x,y) \tag{3}$$

Then $(A, \cdot)$ is a binary loop with the unit e. From

$$\alpha(e,e,z,u) = \alpha(e,e,e,\alpha(e,e,z,u))$$

by condition $D_{3,4}$ we obtain

$$\alpha(x,y,z,u) = \alpha(x,y,e,zu) \tag{4}$$

Now

$$\begin{aligned}\alpha(x,y,e,zu) &= \alpha(e,y,zu) \quad \text{(condition } D_{1-3}\text{)}\\ &= \alpha(e,y,e,x(zu)) \quad \text{(by (4))}\\ &= \alpha(y,e,e,x(zu)) \quad \text{(condition } D_{1,2}\text{)}\\ &= \alpha(e,e,y,x(zu)) \quad \text{(condition } D_{1-3}\text{)}\\ &= y(x(zu)). \quad \text{(by (4))}\end{aligned}$$

Thus

$$\alpha(x,y,z,u) = y(x(zu)) \tag{5}$$

Putting $z = u = e$ in (5) we get

$$\alpha(x,y,e,e) = yx \tag{6}$$

From

$$\alpha(x,y,e,e) = \alpha(\alpha(x,y,e,e),e,e,e)$$

by condition $D_{1,2}$ we have

$$\alpha(x,y,z,u) = \alpha(yx,e,z,u) \tag{7}$$

Now

$$\begin{aligned}\alpha(yx,e,z,u) &= \alpha(e,yx,z,u) \quad \text{(condition } D_{1,2}\text{)}\\ &= \alpha(z,yx,e,u) \quad \text{(condition } D_{1-3}\text{)}\\ &= \alpha((yx)z,e,e,u) \quad \text{(by (7))}\\ &= \alpha((yx)z,e,u,e) \quad \text{(condition } D_{3,4}\text{)}\\ &= \alpha(e,(yx)z,u,e) \quad \text{(condition } D_{1,2}\text{)}\\ &= \alpha(u,(yx)z,e,e) \quad \text{(condition } D_{1-3}\text{)}\\ &= ((yx)z)u \quad \text{(by (7))}\end{aligned}$$

Thus

$$\alpha(x,y,z,u) = ((yx)z)u \tag{8}$$

From (5) and (8) we have

$$y(x(zu)) = (yx)z)u \tag{9}$$

Putting $u = e$ in (9) we obtain

$$y(xz) = (yx)z$$

Therefore $(A, \cdot)$ is a group and $\alpha(x,y,z,u) = yxzu$

The converse is obvious. □

Theorem 2. *The set of all solutions of the functional equation of generalized pseudo-mediality over the set of quasigroup operations on an arbitrary set A is described by the realtions*

$$\begin{array}{ll} \alpha_1(x,y) = F_1(x) + F_2(y) & \alpha_2(x,y) = F_1^{-1}(F_3(y) + F_4(x)) \\ \alpha_3(x,y) = F_2^{-1}(F_5(x) + F_6(y)), & \alpha_4(x,y) = F_7(x) + F_6(y) \\ \alpha_5(x,y) = F_7^{-1}(F_3(y) + F_8(x)), & \alpha_6(x,y) = F_8^{-1}(F_4(x) + F_5(y)) \end{array} \tag{10}$$

where $(A,+)$ is an arbitrary group and $F_1,\dots,F_8$ are arbitrary substitutions of the set A.

Proof. Let $\alpha_1,\dots,\alpha_6$ be six quasigroup operations on A and forming a solution of equation (1). We define $\alpha : A^4 \to A$ by (2). From the above results if follows that $\alpha(x,y,z,u) = T_2(y)+T_1(x)+T_3(z)+T_4(u)$, where $(A,+)$ is a group with zero element $0 = T_1(a_1) = T_2(a_2) = T_3(a_3) = T_4(a_4)$, T_i being translations by $a = (a_1^4) \in A^4$ in (A,α) (for details see [1] and [2]). Putting $z = a_3$ and $u = a_4$ in (2) we get

$$\alpha_1(\alpha_2(x,y), \alpha_3(a_3,a_4)) = \alpha_4(\alpha_5(\alpha_6(x,a_3),y),a_4) = T_2(y) + T_1(x). \tag{11}$$

The mappings $f(x) = \alpha_1(x, \alpha_3(a_3,a_4)), f_1(x) = \alpha_6(x,a_3)$ and $f_2(x) = \alpha_4(x,a_4)$ are substitutions of the set A. From (11) we obtain

$$\alpha_2(x,y) = f^{-1}(T_2(y) + T_1(x))$$

and

$$\alpha_5(x,y) = f_2^{-1}(T_2(y) + T_1(f_1^{-1}(x))).$$

For $x = a_1$ and $y = a_2$ in (2) we have

$$\alpha_1(\alpha_2(a_1,a_2), \alpha_3(z,u)) = \alpha_4(\alpha_5(\alpha_6(a_1,z),a_2),u) = T_3(z) + T_4(u)$$

and thus

$$\alpha_3(z,u) = g^{-1}(T_3(z) + T_4(u))$$

where

$$g(x) = \alpha_1(\alpha_2(a_1,a_2),x)$$

and

$$\alpha_4(z,u) = T_3(g_1^{-1}(z)) + T_4(u)$$

where

$$g_1(x) = \alpha_5(\alpha_6(a_1,x),a_2).$$

Finally, if we put $y = a_2$ and $u = a_4$ in (2) then we have

$$\alpha_1(\alpha_2(x,a_2), \alpha_3(z,a_4)) = \alpha_4(\alpha_5(\alpha_6(x,z),a_2),a_4) = T_1(x) + T_3(z)$$

and thus

$$\alpha_1(x,z) = T_1(h_1^{-1}(x)) + T_3 h_2^{-1}(z)$$

for $h_1(x) = \alpha_2(x,a_2)$ and $h_2(x) = \alpha_3(x,a_4)$,

$$\alpha_6(x,z) = h^{-1}(T_1(x) + T_3(z))$$

where $h(x) = \alpha_4(\alpha_5(x, a_2), a_3)$.

It is easy to prove that

$$f \circ h_1 = T_1, g \circ h_2 = T_3, f_2 \circ g_1 = T_3 \quad \text{and} \quad h \circ f_1 = T_1$$

Taking into account the above results we have

$$\begin{array}{ll} \alpha_1(x,z) = f(x) + g(z), & \alpha_2(x,y) = f^{-1}(T_2(y) + T_1(x)) \\ \alpha_3(z,u) = g^{-1}(T_3(z) + T_4(u)), & \alpha_4(z,u) = f_2(z) + T_4(u) \\ \alpha_5(x,y) = f_2^{-1}(T_2(y) + h(x)), & \alpha_6(x,z) = h^{-1}(T_1(x) + T_3(z)). \end{array}$$

The converse is clear. □

References

[1] Petrescu A., $G-n$-quasigroups, ICTAMI 2007.

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[3] Sokats'kyi F.M., On the classification of functional equations on quasigroups, Ukrain. Math. Journal, Vol. 56, No.9 (2004), 1259-1266.